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SCATTERING OF ELECTROMAGNETIC WAVES BY A CONE-SHAPED

IONIZED WAKE

by

N. P. Mar'yin

(USSR)

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I O N I Z E D   W A K E *

S U M M A R Y

Geometric optics methods are applied to find the trajectories
of rays in a cone-shaped ionized wake, in which the concentration of
free electrons depends on the coordinates r, θ. The variation in the
density of flux's energy in the direction of the ray, having passed
across the ionized medium is also determined without taking into account
the energy in this medium.

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When an ionized point source moves in the atmosphere, there forms

behind it a wake, in which the distribution of the concentration of free

electrons is dependent on two coordinates. The concentration of free elec-

trons gradually decreases along its axis, approaching zero. In cross sec-

tions it has its maximum value on the axis and it decreases toward the

wake's boundary to zero. In order to estimate the influence of free elec-

trons, emerging in the wake under the action of the ionization source, on

the propagation of radiowaves , we may postulate that the variation of

the concentration of free electrons in wake's cross section takes place

proportionally to $(\theta_0^2 - \theta^2)$, where θ_0 is the coordinate defining the boundary

of the wake, and along the wake it takes place proportionally to r^{-2},

where r is the distance from the source to the point considered on wake's

axis. The wake has a shape of a cone with the source at its summit. In

order to simplify the calculations, we shall discount in the following

the source's motion; in other words, we shall determine the trajectories

of rays in a fixed ionized cone with the above-indicated concentration of

free electrons.

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* RASSEYANIYE LUCHEY ELEKTROMAGNITNYKH VOLN KONUSOOBRAZNYM IONIZIROVANNYM
SLEDOM.

The choice of the dependence of the concentration of free electrons along the axis of the wake in the form r^{-2} is determined by the condition of separation of curvilinear coordinates in the eikonal equation. In the wake's cross section the choice of dependence on θ of electron concentration may be rather arbitrary.
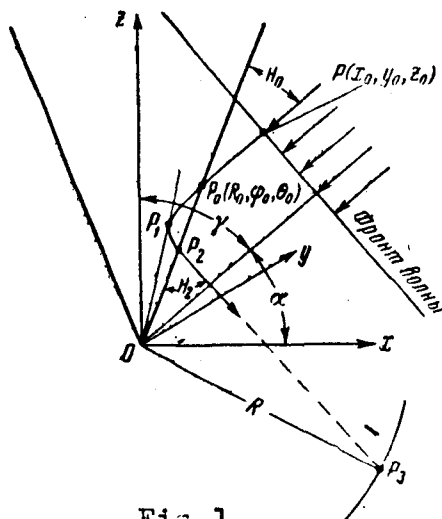

Fig. 1

#1. - TRAJECTORIES OF RAYS.

In a spherical system of coordinates (r, φ, θ), of which the origin coincides with the summit of the cone and the axis Oz is directed along the axis of the wake (Fig. 1), the distribution of electron concentration has the form

$$N(r, \varphi, \theta) = N_0 \frac{r_0^2}{\theta_0^2} \frac{\theta_0^2 - \theta^2}{r^2}, \qquad (1)$$

where θ_0 is the coordinate of wake's boundary. Since the wake is axisymmetric $N(r, \varphi, \theta)$ does not depend on φ, that is, $N(r, \varphi, \theta) = N(r, \theta)$.

The relative dielectric constant of the ionized wake is

$$\varepsilon_r = 1 - \frac{4\pi e^2 N_0 (\theta_0^2 - \theta^2)}{m(\omega^2 + v^2) r^2} \frac{r_0^2}{\theta_0^2}. \qquad (2)$$

Within the bounds of geometric optics the motion of an electromagnetic wave in an ionized wake is determined with the help of a family of equiphasal surfaces perpendicular to the rays. These surfaces $L = $ const

satisfy the nonlinear differential equation [1, 2]

$$(\nabla L)^2 = n^2, \quad n^2 = \varepsilon_r. \tag{3}$$

In the orthogonal curvilinear system of coordinates (q_1, q_2, q_3) the eikonal equation may be represented in the form

$$\left(\frac{1}{h_1}\frac{\partial L}{\partial q_1}\right)^2 + \left(\frac{1}{h_2}\frac{\partial L}{\partial q_2}\right)^2 + \left(\frac{1}{h_3}\frac{\partial L}{\partial q_3}\right)^2 = n^2, \tag{4}$$

where h_1, h_2, h_3 are Lamé coefficients.

In spherical coordinates, (3), taking into account (2), can be written as follows :

$$\left(\frac{\partial L}{\partial r}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial L}{\partial \varphi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial L}{\partial \theta}\right)^2 = 1 - \frac{b^2(\theta_0^2 - \theta^2)}{r^2},$$

where

$$b^2 = \frac{4\pi e^2 N_0}{m(\omega^2 + \nu^2)}\frac{r_0^2}{\theta_0^2}. \tag{5}$$

The general integral of (5) may be constructed by the method of variable separation, postulating

$$L = f_1(r) + f_2(\varphi) + f_3(\theta). \tag{6}$$

Integrating, we shall have

$$f_1(r) = \int_{R_0}^{r} \sqrt{1 - a_1^2/r_1^2}\, dr, \quad f_2(\varphi) = a_2\varphi + a_3,$$

$$f_3(\theta) = \int_{\theta_0}^{\theta} \sqrt{a_1^2 - b^2(\theta_0^2 - \theta^2) - a_2^2/\sin^2\theta}\, d\theta. \tag{7}$$

The total integral of (5) is equal to the sum of integrals (7), which depend on three constants a_1, a_2, a_3; at the same time from a_3 additively

$$L = \psi(r, \varphi, \theta, a_1, a_2) + a_3. \tag{8}$$

According to the Jacobi theorem [2], when the solution of the eikonal equation is known, the corresponding family of rays is determined by the equalities

$$\frac{\partial}{\partial a_i}\psi(r, \varphi, \theta, a_1, a_2) = C_i \quad (i = 1, 2), \tag{9}$$

where a_1, a_2, C_1 and C_2 are arbitrary constants, defining any two points

through which the ray passes, or the point crossed by the ray in the given direction.

Taking into account (7) and effecting the differentiation of (9), we we shall obtain a system of equations determining the corresponding family of rays

$$\int_{R_0}^{r} \frac{a_1 dr}{r\sqrt{r^2-a_1^2}} - \int_{\theta_0}^{\theta} \frac{a_1\sin\theta\,d\theta}{\sqrt{[a_1^2-b^2(\theta_0^2-\theta^2)]\sin^2\theta-a_2^2}} = C_1,$$

$$\varphi - \int_{\theta_0}^{\theta} \frac{a_2 d\theta}{\sin\theta\sqrt{[a_1^2-b^2(\theta_0^2-\theta^2)]\sin^2\theta-a_2^2}} = C_2.$$

We shall select the constants C_1 and C_2 from the condition that the point R_0, φ_0, θ_0 lay on the ray. Consequently,

$$\int_{R_0}^{r} \frac{dr}{r\sqrt{r^2-a_1^2}} = \int_{\theta_0}^{\theta} \frac{\sin\theta\,d\theta}{\sqrt{[a_1^2-b^2(\theta_0^2-\theta^2)]\sin^2\theta-a_2^2}},$$

$$\varphi - \int_{\theta_0}^{\theta} \frac{a_2 d\theta}{\sin\theta\sqrt{[a_1^2-b^2(\theta_0^2-0^2)]\sin^2\theta-a_2^2}} = \tilde{\varphi}_0, \qquad (10)$$

or

$$F^{(1)} = 0, \qquad F^{(2)} = 0.$$

The system of equations (10) defines the trajectories of the rays passing through the point $P_0(R_0, \varphi_0, \theta_0)$ The direction of the ray's motion at that point is given by the constants a_1 and a_2. In order to determine the constants a_1 and a_2, it is necessary to assign the plane wave front equation for the wave incident upon the ionized wake, or that of an arbitrary ray normal to that plane. Of the aggregate of rays defining the plane wave the remaining ones will be collinear with them. The plane, normal to the central ray, is the wave front plane. It is always possible so to select the system of coordinates that the axis of the wake and the central ray lay in the plane **xoz**. Then the axis **oy** will be parallel to the wave front and the front plane equation may be written in the form

$$Ax + Cz + D = 0. \qquad (11)$$

The equation for the ray passing through the point $P(x_0, y_0, z_0)$ lying in the plane of the wave front, may be written as follows :

$$(x-x_0)/A = (z-z_0)/C, \quad y = y_0.$$

The intersection of this ray with the conical surface of the wake, having for equation

$$z^2 \, \mathrm{tg}^2 \, \theta_0 = x^2 + y^2,$$

will take place at the point P_0, of which the coordinates are determined from the system of equations

$$(x - x_0)/A = (z - z_0)/C, \quad z^2 \, \mathrm{tg}^2 \, \theta_0 = x^2 + y^2, \quad y = y_0.$$

Resolving the system of equations and passing to spherical coordinates, we shall obtain

$$R_0 = A_1 + \sqrt{A_1^2 + B_1^2}, \quad \varphi_0 = \arcsin \frac{y_0}{R_0 \sin \theta_0} \quad, \quad \theta_0 = \theta_0,$$

$$\mathrm{tg} \, \gamma = \frac{A}{C}, \quad A_1 = \frac{\mathrm{tg} \, \gamma \cos \theta_0 (x_0 - z_0 \, \mathrm{tg} \, \gamma)}{(1 - \mathrm{tg}^2 \gamma \cos^2 \theta_0)}, \quad B_1 = \frac{\dfrac{y_0^2}{\sin^2 \theta_0} + (x_0 - z_0 \, \mathrm{tg} \, \gamma)^2}{(1 - \mathrm{tg}^2 \gamma \cos^2 \theta_0)}.$$

Thus, we have found the coordinates of the point at wake's surface, from which the ray propagates inside the wake in a nonuniform medium.

The **vector T** equation in a spherical system of coordinates, when this vector is tangent to ray trajectory at an arbitrary point inside the the wake, is then determined as the vectorial product of function's $F^{(1)}$ gradient

$$\nabla F^{(1)} = \frac{\partial F^{(1)}}{\partial r} e_r + \frac{1}{r} \frac{\partial F^{(1)}}{\partial \theta} e_\theta,$$

and function's $F^{(2)}$ gradient

$$\nabla F^{(2)} = \frac{\partial F^{(2)}}{\partial r} e_r + \frac{1}{r} \frac{\partial F^{(2)}}{\partial \theta} e_\theta.$$

Consequently,

$$\mathbf{T} = - \frac{\partial F^{(1)}}{\partial \theta} \frac{\partial F^{(2)}}{\partial \varphi} e_r - r \sin \theta \frac{\partial F^{(1)}}{\partial r} \frac{\partial F^{(2)}}{\partial \theta} e_\varphi + r \frac{\partial F^{(1)}}{\partial r} \frac{\partial F^{(2)}}{\partial \varphi} e_\theta. \quad (12)$$

At the point P_0 the vector **T** must be collinear with the incident ray, of which the unitary vector in the spherical system of coordinates may be represented in the form

$$s_0 = - (\cos \alpha \sin \theta \cos \varphi + \cos \gamma \cos \theta) e_r + \cos \alpha \sin \varphi \, e_\varphi -$$
$$- (\cos \alpha \cos \varphi \cos \theta - \cos \gamma \sin \theta) e_\theta, \quad (13)$$

where

$$\alpha = \mathrm{arc} \, \mathrm{tg} \, C/A.$$

The vectors of T and s_0 at the point P_0 are collinear if

$$-r \frac{\partial F^{(1)}/\partial r}{\partial F^{(1)}/\partial \theta} \bigg|_{P_0} = \frac{\cos \alpha \cos \varphi_0 \cos \theta_0' - \cos \gamma \sin \theta_0}{\cos \alpha \cos \varphi \sin \theta_0 + \cos \gamma \cos \theta_0}.$$

$$-\sin \theta_0 \frac{\partial F^{(2)}/\partial \theta}{\partial F^{(2)}/\partial \varphi} \bigg|_{P_0} = \frac{\cos \alpha \sin \varphi_0}{(\cos \alpha \cos \varphi_0 \cos \theta_0 - \cos \gamma \sin \theta_0)}. \qquad (14)$$

Substituting in (14)

$$\frac{\partial F^{(1)}}{\partial r}, \ \frac{\partial F^{(1)}}{\partial \theta}, \ \frac{\partial F^{(2)}}{\partial \theta}, \ \frac{\partial F^{(2)}}{\partial \varphi},$$

determined on the basis of (1), we shall obtain

$$\frac{\sqrt{a_1^2 \sin^2 \theta_0 - a_2^2}}{\sqrt{R_0^2 - a_1^2 \sin \theta_0}} = \frac{\cos \alpha \cos \varphi_0 \cos \theta_0 - \cos \gamma \sin \theta_0}{\cos \alpha \cos \varphi_0 \cos \theta_0 + \cos \gamma \cos \theta_0}$$

$$\frac{a_2}{\sqrt{a_1^2 \sin^2 \theta_0 - a_2^2}} = \frac{\cos \alpha \sin \varphi_0}{\cos \alpha \cos \varphi_0 \cos \theta_0 - \cos \gamma \sin \theta_0} \qquad (15)$$

Resolving the system of equations (15) relative to unknowns a_1 and a_2, we shall have

$$a_1^2 = R_0^2 [\cos^2 \alpha \sin^2 \varphi_0 + (\cos \alpha \cos \varphi_0 \cos \theta_0 - \cos \gamma \sin \theta_0)^2] / E, \quad (16)$$

$$a_2^2 = R_0^2 \cos^2 \alpha \sin^2 \varphi_0 \sin^2 \theta^2 / E, \qquad (17)$$

where

$$E = (\cos \alpha \cos \varphi_0 \cos \theta_0 + \cos \gamma \cos \theta_0)^2 + \cos^2 \alpha \sin^2 \varphi_0 +$$
$$+ (\cos \alpha \cos \varphi_0 \cos \theta_0 - \cos \gamma \sin \theta_0)^2.$$

Let us study the trajectories of rays incident upon the ionized wake in the plane $\varphi = \tilde{\varphi}_0 = 0$ at further length. In this case, first of all, at the above made observations on the system of coordinate selection

$$\alpha + \gamma = \pi/2 \qquad (18)$$

and, secondly, according to (16) and (17), the quantities a_1 and a_2 are respectively equal to

$$a_1 = R_0 \sin (\gamma - \theta_0), \ a_2 = 0. \qquad (19)$$

Consequently, the trajectories of rays in the wake are determined by the first equation of the system (10).

$$\int_{R_0}^{r} dr/r \sqrt{r^2 - a_1^2} = \int_{\theta_0}^{\theta} d\theta/\sqrt{a_1^2 - b^2(\theta_0^2 - \theta^2)} \qquad (20)$$

and they too lie in the plane $\varphi = \tilde{\varphi}_0 = 0$.

The tangent of the angle between the tangent to the ray at the arbitrary point inside the wake and the radius-vector r may be determined from (14), but in the given case it is simpler to find it from (20):

$$\operatorname{tg} H = r\frac{d\theta}{dr} = \sqrt{a_1^2 - b^2(\theta_0^2 - \theta^2)}/\sqrt{r^4 - a_1^2}. \tag{21}$$

Analysis of (21) shows that when the tangent to the ray coincides with the radius-vector ($\operatorname{tg} H = 0$), the condition

$$a_1^2 = b^2(\theta_0^2 - \theta^2), \tag{22}$$

must be satisfied; this condition determines the coordinate of the point of ray's rotation in the wake as a function of the concentration of free electrons in it and of its geometric dimensions. It follows from (22) that the rotation of the ray takes place at wake boundary $\theta = \theta_0$, when the concentration of free electrons on wake's boundary increases to infinity ($b^2 \to \infty$). The radius-vector of the point at which ray's rotation takes place at $b^2 \to \infty$ is, according to (20), equal to R_0. Consequently, the reflection of the ray from the boundary of the wake takes place in this case as if it were from a metallic surface. If $b^2 < \infty$, the ray's rotation takes place at the point of which the coordinate θ_1 is determined by the equation

$$\theta_1^2 = \theta_0^2 - a_1^2/b^2. \tag{23}$$

The coordinate r_1 of this point is determined after integrating (20) and substituting (23) in the expression obtained after the integration

$$r_1 = a_1\left\{\cos\left[\arccos a_1/R_0 + \frac{a_1}{2b}\ln\frac{\theta_0 b - a_1}{\theta_0 b + a_1}\right]\right\}^{-1}. \tag{24}$$

When $\theta_1 = 0$, the ray is reflected from the axis of the wake and the condition (22) has the form

$$a_1^2 = b^2\theta_0^2 \quad \text{or} \quad R_0^2\sin^2(\gamma - \theta_0) = b^2\theta_0^2. \tag{25}$$

Therefore, for the given concentrations of free electrons and wake widening (angle θ_0), the rays will pass through the wake only at quite specific values of R_0 and γ. The rays having crossed the wake undergo distortion. As they drift away from wake's pole, the distortion of the rays decreases, which is easy to establish according to (21) and taking into account (20).

The expression (21) allows to find the angle H between the radius-vector and the ray at the point of ray's incidence on the boundary of the wake as well as at the point $P_1(R_2, 0, \theta_0)$, where the ray crosses the wake's boundary when emerging from it. Integrating (20), we find the value of R_2

$$R_2 = a_1 \left\{ \cos \left[\arccos a_1/R_0 + \frac{a_1}{b} \ln \frac{\theta_0 b - a_1}{\theta_0 b + a_1} \right] \right\}^{-1} \qquad (26)$$

Substituting (26) into (21), we shall determine the angle H_2 at the point P_2

$$H_2 = H_0 - \frac{a_1}{b} \ln \frac{(\theta_0 b - a_1)}{(\theta_0 b + a_1)}, \qquad (27)$$

where H_0 is the angle between the radius-vector and the ray at incidence point.

It follows from (27) that $|H_2| \neq |H_0|$. When the concentration of free electrons on the axis of the wake increases, the angle $|H_2| = |H_0|$ and within the bounds $|H_2| \rightarrow |H_0|$ at $b^2 \rightarrow \infty$. If the incident ray propagates along the wake's boundary so that $H = \sin(\gamma - \theta_0) = 0$, $H_2 = H_0$ in that case too. This means that the ray propagates rectilinearly.

Finally, let us find the condition at which the radioray, incident upon the wake, is reflected strictly backward. Formula (21) defines the angle between the tangent to ray trajectory and the radius-vector of the point. At the point P_2 we must have $H_2 = -H_0 + \pi$.

Substituting H_2 in (27), we obtain

$$\exp \left[\frac{(\pi - 2H_0) b}{a_1} \right] = \frac{\theta_0 b + a_1}{\theta_0 b - a_1}, \qquad (28)$$

with, at the same time $R_2 = R_0$. This equality is fulfilled when the ray propagates in an ionized medium *from greater* values of r to smaller ones.

If the incident ray propagates perpendicularly to cone's generatrix, in the plane xoz (angle $H_0 = \pi/2$), the angle H_2 will be smaller than $\pi/2$. We shall have $H_2 = -H_0 + \pi$ only when the equality (28) is fulfilled. That is why (28) is the condition for the mirror reflection of the ray from the ionized wake, in which the concentration of free electrons varies according to the law (1). In this case the value of H_0 is determined from the equation

../..

$$\exp\left[\frac{(\pi - 2H_0)b}{R_0 \sin H_0}\right] = \frac{\theta_0 b/R_0 + \sin H_0}{\theta_0 b/R_0 - \sin H_0}. \qquad (29)$$

Therefore, when the wake is investigated by radar, it may be detected only at irradiation from directions defined by the angles of H_0 according to (29). The wake may be observed in the spherical circle defined by $\theta_0 \leqslant \gamma \leqslant \pi/2 + \theta_0$.

As an example, let us find the points of reflection of rays, coplanar with the plane **xoz** . To that effect we shall determine the angle H between the vector **T**, tangent to ray trajectory at an arbitrary point inside the wake and the coordinate plane in which lie the orts e_r and e_φ. It follows from (12) and (10) that

$$\operatorname{tg} H = \sqrt{\{[a_1{}^2 - b^2(\theta_0{}^2 - \theta^2)]\sin^2\theta - a_2{}^2\} / [\sin^2\theta(r^2 - a_1{}^2) - a_2{}^2]}. \qquad (30)$$

When $\operatorname{tg} H = 0$, the vector **T** lies in the coordinate plane of orts and e_φ; from that moment the ray turns and begins to emerge from the wake. The coordinate θ_1 of the rotation point may be found from the expression

$$[a_1{}^2 - b^2(\theta_0{}^2 - \theta_1{}^2)]\sin\theta_1 - a_2{}^2 = 0,$$

which follows from (30). At $\theta_1 \leqslant \theta_0 \ll \pi/2$

$$\theta_1{}^2 \approx \frac{1}{2b^2}(\theta_0{}^2 b^2 - a_1{}^2)[1 + \sqrt{1 + 4a_2{}^2 b^2/(\theta_0{}^2 b^2 - a_1{}^2)}]. \qquad (31)$$

Knowing θ_1, we determine the remaining two coordinates, that is, those of the ray rotation (r_1, φ_1) according to equations (10). It is then necessary to bear in mind that in (10) we have $\widetilde{\varphi}_0 = 0$ at the expense of the choice of coordinate system. Since the angle θ_0 is small, the integrals in (10) may be taken provided we postulate $\sin\theta \approx \theta$. Then

$$\int_{R_0}^{r}\frac{dr}{r\sqrt{r^2 - a_1{}^2}} = \int_{\theta_0}^{\theta}\frac{\theta d\theta}{\sqrt{[a_1{}^2 - b^2(\theta_0{}^2 - \theta^2)]\theta^2 - a_2{}^2}},$$

$$\varphi = \int_{\theta_0}^{\theta}\frac{a_2 d\theta}{\theta\sqrt{[a_1{}^2 - b^2(\theta_0{}^2 - \theta^2)]\theta^2 - a_2{}^2}}.$$

$$r_1 = \frac{a_1}{\cos B_1}, \qquad B_1 = \arccos\frac{a_1}{R_0} - \frac{a_1}{2b}\ln\frac{2b^2\theta_1{}^2 + a_1{}^2 - b^2\theta_0{}^2}{2b\sqrt{a_1{}^2\theta_0{}^2 - a_2{}^2} + \theta_0{}^2 b^2 + a_1{}^2}.$$

Upon the substitution of the variable $\theta^2 = \xi$, the second integral may be written in the form

$$\varphi_1 = \int_{\theta_0^2}^{\theta^2} \frac{a_2 \, d\xi}{2\xi \, \sqrt{[a_1^2 - b^2(\theta_0^2 - \xi)]\xi - a_2^2}} =$$

$$= \frac{1}{2}\left[\arcsin \frac{2a_2^2 + (\theta_0^2 b^2 - a_1^2)\theta_0^2}{\theta_0^2 \sqrt{(a_1^2 - \theta_0^2 b^2)^2 + 4a_2^2 b^2}} - \arcsin \frac{2a_2^2 + (\theta_0^2 b^2 - a_1^2)\theta_1^2}{\theta_1^2 \sqrt{(a_1^2 - \theta_0^2 b^2)^2 + 4a_2^2 b^2}} \right]. \quad (33)$$

Therefore, the expressions (31), (32) and (33) define the coordinates of the rotation point of the ray $P_1(r_1, \varphi_1, \theta_1)$. After the rotation point the ray will pass to the boundary of the wake emerging from it. The point of wake surface's intersection by the ray $P_2(r_2, \varphi_2, \theta_0)$ is determined by the system of equations (10) and the equation $\theta = \theta_0$. Taking into account the direction of integration in (10), and after fairly simple transformations, we shall obtain the coordinates of the point P_2

$$r_2 = \frac{a_1}{\cos B_2}, \quad (34)$$

$$B_2 = \arccos \frac{a_1}{R_0} - \frac{a_1}{b} \ln \frac{2b^2\theta_1^2 + a_1^2 - \theta_0^2 b^2}{(2b\sqrt{a_1^2\theta_0^2 - a_2^2} + \theta_0^2 b^2 + a_1^2)}$$

$$\varphi_2 = 2\varphi_1, \quad \theta_2 = \theta_0. \quad (35)$$

After the point P_2, the ray is situated beyond the wake and propagates rectilinearly in the direction of the vector $\mathbf{T}$, defined at the point P_2.

#2. — DENSITY VARIATION OF ENERGY FLUX IN THE PROPAGATION DIRECTION OF THE REFLECTED RAY

The ray emerges from the wake at the point P_2. The coordinates of this point in the rectangular system of coordinates are

$$x_2 = r_2 \cos \varphi_2 \cos \theta_0, \quad y_2 = r_2 \cos \varphi_2 \sin \theta_0, \quad z_2 = r_2 \cos \theta_0.$$

The ort projection of the ray at the point P_2 may be written in the rectangular system of coordinates in the form

$$T_x = T_r \sin \theta_0 \cos \varphi_2 - T_\varphi \sin \varphi_2 + T_\theta \cos \varphi_2 \cos \theta_0,$$
$$T_y = T_r \cos \varphi_2 + T_\varphi \cos \varphi_2 + T_\theta \sin \varphi_2 \cos \theta_0,$$
$$T_z = T_r \cos \theta_0 - T_\theta \sin \theta_0.$$

Since T_0 is a unitary vector, T_x, T_y, T_z are equal to the direction cosines of the vector T_0 in the rectangular system of coordinates. The equation of the line passing through the point P_2 parallelwise to the directing vector T_0 will be represented in the form

$$\frac{x-x_2}{T_x} = \frac{y-y_2}{T_y} = \frac{z-z_0}{T_z}.$$

This line will cross the sphere having the center at coordinate origin at the points P_3, P_4, of which the coordinates are determined by the system of equations

$$\frac{x-x_2}{T_x} = \frac{y-y_2}{T_y} = \frac{z-z_2}{T_z}, \qquad x^2+y^2+z^2 = R^2.$$

One of these points is the point of observations. Its coordinates are

$$x_3 = -T_r T_x r_2 + [T_r^2 T_x^2 r_2^2 + T_x^2 R^2 - T_\varphi^2 r_2^2 \sin^2\theta_0 +$$
$$+ r_2^2(T_\varphi \sin\varphi_2 \cos\theta_0 + T_\theta \cos\varphi_2)^2 + x_2]^{1/2},$$
$$y_3 = (T_y/T_x)(x_3-x_2) + y_2, \quad z_3 = (T_z/T_x)(x_3-x_2) + z_2.$$

Since the coordinates of the point P_3 are known, the radius-vector of this point in the vectorial form is

$$\bar{R} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}.$$

The elementary area of the receiving device is situated at the point P_3; it is so oriented that the direction of the vector normal to its surface coincides with the direction of $\mathbf{R}$.

The flux of energy through the elementary area ds_0, cut out on the surface of the incident wave front, remains constant within the limits of the tube formed by the rays, passing along the boundary of the elementary area, provided we neglect the losses of electromagnetic energy in the ionized medium. The energy flux through ds_0 is

$$S_0 ds_0 = S_0 \frac{dz_0\, dy_0}{\cos\alpha},$$

where it is taken into account that the density vector of energy S_0 flux is collinear to the vector of the elementary area ds_0.

The second elementary area is formed as a result of intersection

of the tube of rays by a surface perpendicular to vector **R** (sphere).

If in the plane of the incident wave front the ray is displaced in the direction of the axis <u>oz</u>, the ray reflected from the wake will also have its direction changed. That is why the vector **R** will have an accretion $(d\mathbf{R}/dz_0)dz_0,$ provided the incident ray shifts in the front plane along the direction of the axis <u>oz</u> * The elementary area ds at the point of observation may be determined by the formula

$$ds_3 = \left[\frac{d\mathbf{R}}{dz_0}dz_0, \quad \frac{d\mathbf{R}}{dy_0}dy_0\right].$$

The flux of energy through that area is

$$Sds_3 = ST_0\left[\frac{d\mathbf{R}}{dz_0}dz_0, \quad \frac{d\mathbf{R}}{dy_0}dy_0\right].$$

According to the law of energy conservation $S_0ds_0 = Sds_3$. Hence, the energy flux density at the point of observation is

$$S = S_0/\cos\alpha\, T_0\left[\frac{d\mathbf{R}}{dz_0}, \quad \frac{d\mathbf{R}}{dy_0}\right]. \tag{36}$$

Consequently, formula (36) allows to determine the density of energy flux for every ray having crossed the ionized wake, provided we neglect the energy absorption along its trajectory.

In order to establish the direct dependence of the energy flux density on the direction of ray incidence, on wake geometry and on the coordinates of the point of observation, it is necessary to transform the scalar-vectorial expression entering in (36). Inasmuch as the module of **R** is a constant quantity, we have

$$S = S_0/R^2\cos\alpha\, T_0\,[dr_0/dz_0, dr_0/dy_0], \tag{37}$$

where

$$\mathbf{r}_0 - \mathrm{opr}\,\mathbf{R}, \quad \mathbf{r}_0 = x_{03}\mathbf{i} + y_{03}\mathbf{j} + z_{03}\mathbf{k}, \quad x_{03} = \frac{x_3}{R}, \quad y_{03} = \frac{y_3}{R}, \quad z_{03} = \frac{z_3}{R} \tag{38}$$

Expressing $\mathbf{r}_0$ and $\mathbf{T}_0$ through their projections on the axis of the rectangular system of coordinates, we shall obtain

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* Analogously **R** will have will have an accretion $(d\mathbf{R}/dy_0)dy_0,$ if the incident ray shifts from the front plane in the dircetion of the axis <u>oy</u>.

$$S = \frac{S_0}{R^2 \cos \alpha}\Big[\Big(\frac{\partial y_{03}}{\partial y_0}\frac{\partial z_{03}}{\partial z_0} - \frac{\partial y_{03}}{\partial z_0}\frac{\partial z_{03}}{\partial y_0}\Big) + \Big(\frac{\partial z_{03}}{\partial z_0}\frac{\partial x_{03}}{\partial z_0} - \frac{\partial x_{03}}{\partial z_0}\frac{\partial z_{03}}{\partial y_0}\Big) +$$

$$+ \Big(\frac{\partial y_{03}}{\partial y_0}\frac{\partial x_{03}}{\partial z_0} - \frac{\partial y_{03}}{\partial z_0}\frac{\partial x_{03}}{\partial y_0}\Big)\Big]^{-1}. \tag{39}$$

Thus, formula (39) allows us to compute the energy flux density in the direction of the vector **T**, tangent to ray trajectory at the point P_2.

Let us find, as an example, the re-emission pattern of the wave plane by the wake in the plane $\tilde{\varphi}_2 = 0$ at radar irradiation of the wake. It will reflect toward the side of the radar station provided the condition (28) is satisfied. In this case the density of energy flux in the direction toward the radar station may be found from the formula

$$S \approx \frac{S_0 R_0^2 \sin 2H_0}{2R^2 \cos^3 \alpha}\Big[(\pi - 2H_0)\cos H_0 + \sin H_0 + \frac{a_1^2 2\theta_0 \cos H_0}{\theta_0^2 b^2 - a_1^2}\Big]^{-1} \times$$

$$\times \Big[\frac{2\sin^2 H_0 + 1}{2} + \frac{2a_1 R_0 \sin(H_0 + \theta_0)}{\theta_0(b^2\theta_0^2 - a_1^2)}\Big]^{-1} \tag{40}$$

Since the quantity $a_1 = R_0 \sin H_0 = \text{const}$, there may be, for every selected ray having $R_o = R_{Oi}$, only one direction H_{Oi}, in which the wake may be detected by radar. But from the conjunction of rays forming a plane wave, we may select R_{Ok} in the sector $+\theta_0 \leqslant \gamma \leqslant \pi/2 + \theta_0$ $(0 \leqslant H_0 \leqslant \pi/2)$ in such a way, that the wake reflects in the direction H_{Ok} strictly backward. The density of energy flux will in each case be determined by (40), where $r_2 \ll R, \theta_0 \ll \pi, b$ is limited $(b < \infty)$. When $b \to \infty$, the radar detection of the wake is possible at $\sin H_0 \to \pi/2$. If b decreases, the rays are not reflected from the wake at $\theta_0^2 b^2 - a^2 \leqslant 0$.

*** T H E   E N D ***

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